第四节　求不定积分的方法

1、配凑法

例 1、求 $\int 2\cos 2xdx$

解：$\int 2\cos 2xdx = \int \cos 2xd2x = \int d\sin 2x = \sin 2x + C$

例 2、求 $\int \frac{1}{3+2x}dx$

解：$\int \frac{1}{3+2x}dx = \frac{1}{2}\int \frac{1}{3+2x}d(3+2x) = \frac{1}{2}\ln|3+2x| + C$

例 3、求 $\int \sqrt{2x+1}dx$

解：$\int \sqrt{2x+1}dx = \frac{1}{2}\int (2x+1)^{\frac{1}{2}}d(2x+1) = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$

例 4、求 $\int (2xe^{x^2} + x\sqrt{1-x^2} + \tan x)dx$

解：原式$= \int 2xe^{x^2}dx + \int x\sqrt{1-x^2}dx + \int \frac{\sin x}{\cos x}dx$

$$= \int e^{x^2}dx^2 - \frac{1}{2}\int (1-x^2)^{\frac{1}{2}}d(1-x^2) - \int \frac{1}{\cos x}d\cos x$$

$$= e^{x^2} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} - \ln|\cos x| + C$$

例 5、求 $\int [\frac{1}{x(1+2\ln x)} + \frac{1}{\sqrt{x}}e^{3\sqrt{x}}]dx$

解：$\int [\frac{1}{x(1+2\ln x)} + \frac{1}{\sqrt{x}}e^{3\sqrt{x}}]dx = \int \frac{1}{x(1+2\ln x)}dx + \int \frac{1}{\sqrt{x}}e^{3\sqrt{x}}dx$

$$= \frac{1}{2}\int \frac{1}{1+2\ln x}d(1+2\ln x) + \frac{2}{3}\int e^{3\sqrt{x}}d3\sqrt{x} = \frac{1}{2}\ln|1+2\ln x| + \frac{2}{3}e^{3\sqrt{x}} + C$$

例 6、求 $\int \cos^2 xdx$

解：$\int \cos^2 xdx = \int \frac{1+\cos 2x}{2}dx = \frac{1}{2}[\int dx + \int \cos 2xdx]$

$$= \frac{x}{2} + \frac{1}{4}\int \cos 2xd2x = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

例 7、求 $\int \frac{dx}{x^2+2x+3}$

解:$\int \frac{dx}{x^2+2x+3} = \int \frac{1}{x^2+2x+1+2}dx = \int \frac{1}{(x+1)^2+(\sqrt{2})^2}d(x+1) = \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$

2、换元法

例 1、求 $\int\sqrt{2x+1}dx$

解：设 $t=\sqrt{2x+1}$,则 $x=\frac{t^2-1}{2}, dx=tdt, \int\sqrt{2x+1}dx=\int t^2dt=\frac{1}{3}t^3+C=\frac{1}{3}(2x+1)^{\frac{3}{2}}+C$

例 2、求 $\int x\sqrt{1-x^2}dx$

解：$t=1-x^2$,则 $dt=-2xdx=-\frac{1}{2}dt, \int x\sqrt{1-x^2}dx=-\frac{1}{2}\int\sqrt{t}dt=-\frac{3}{4}t^{\frac{3}{2}}+c=-\frac{3}{4}\sqrt{(1-x^2)^3}+c_1$

例 3、求 $\int\sqrt{a^2-x^2}dx$，$(a>0)$

解：令 $x=a\sin t$，$-\frac{\pi}{2}\le t\le\frac{\pi}{2}$，则

$\sqrt{a^2-x^2}=a\cos t$，$dx=a\cos tdt$，因此有

$$\begin{aligned}\int\sqrt{a^2-x^2}dx&=\int a\cos t a\cos tdt=a^2\int\cos^2 tdt=a^2\int\frac{1+\cos 2t}{2}dt\\&=\frac{a^2}{2}t+\frac{a^2}{4}\sin 2t+C=\frac{a^2}{2}t+\frac{a^2}{2}\sin t\cos t+C=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{a^2}{2}\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}+C\\&=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C\end{aligned}$$

例 4、求 $\int\frac{dx}{\sqrt{a^2+x^2}}$，$(a>0)$

解：令 $x=a\tan t$，$-\frac{\pi}{2}\le t\le\frac{\pi}{2}$，则 $\sqrt{a^2+x^2}=a\sec t$，$dx=a\sec^2 tdt$，因此有

$$\begin{aligned}\int\frac{dx}{\sqrt{a^2+x^2}}&=\int\frac{1}{a\sec t}a\sec^2 tdt=\int\sec tdt\\&=\ln|\sec t+\tan t|+C=ln|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}|+C=\ln|x+\sqrt{x^2+a^2}|+C_1\end{aligned}$$

3、分部积分

$$\int u(x)dv(x)+\int v(x)du(x)=\int[u(x)dv(x)+v(x)du(x)]$$
$$=\int d[u(x)v(x)]=u(x)v(x)+c$$

因此 $\int u(x)dv(x)=u(x)v(x)-\int v(x)du(x)$，这就是分部积分公式

例 1、求 $\int x\cos xdx$

解： $\int x\cos xdx=\int xd\sin x=x\sin x-\int\sin xdx=x\sin x+\cos x+C$

例 2、求 $\int x^2 e^x dx$

解: $\int x^2 e^x dx = \int x^2 de^x = x^2 e^x - \int e^x dx^2 = x^2 e^x - 2\int e^x dx = x^2 e^x - 2\int xde^x$

$= x^2 e^x - 2xe^x + 2\int e^x dx = x^2 e^x - 2xe^x + e^x + C$

例 3、求 $\int x \ln x dx$

解: $\int x \ln x dx = \frac{1}{2}\int \ln x dx^2 = \frac{1}{2}\left[x^2 \ln x - \int x^2 d\ln x\right] = \frac{1}{2}\left[x^2 \ln x - \int x dx\right]$

$= \frac{1}{2}\left[x^2 \ln x - \frac{1}{2}x^2\right] + C = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

例 4、求 $\int x \arctan x dx$

解: $\int x \arctan x dx = \frac{1}{2}\int \arctan x dx^2 = \frac{1}{2}\left[x^2 \arctan x - \int x^2 d \arctan x\right]$

$= \frac{1}{2}\left[x^2 \arctan x - \int \frac{x^2}{1+x^2} dx\right] = \frac{1}{2}\left[x^2 \arctan x - \int (1 - \frac{1}{1+x^2}) dx\right]$

$= \frac{1}{2}\left[x^2 \arctan x - x + \arctan x\right] + C$

例 5、 求 $\int e^x \sin x dx$

解： $\int e^x \sin x dx = -\int e^x d\cos x = -e^x \cos x + \int \cos x e^x dx$

$\int e^x \cos x dx = -\int e^x d\sin x = -e^x \sin x + \int e^x \sin x dx$

可联立解得 $\int e^x \sin x dx$ 与 $\int e^x \cos x dx$

4、分式积分

一个简分式 $\frac{P(x)}{Q(x)}$ (分子的次数小于分母的次数)，当分母 $Q(x)$ 分解成一次与二次不可约因式之积时，例如 $Q(x) = (x-1)(x^2+x+1)(x-2)^3(x^2-x+1)^2$ 时，则，分式可化为部分分式的和：$\frac{P(x)}{Q(x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{F}{(x-2)^3} + \frac{Gx+H}{x^2-x+1} + \frac{Ix+J}{(x^2-x+1)^2}$

注意到：$\int \frac{1}{Ax+B} d = \frac{1}{A}\ln|Ax+B|, \int \frac{1}{(Ax+B)^n} dx = -\frac{1}{A(n+1)}(Ax+B)^{n+1}$

$\int \frac{1}{x^2+Q^2} dx = Qarc\tan\frac{x}{Q}$，就可求所有分式的不定积分了

例 1．求 $\int \frac{1+x+x^2}{x(1+x^2)}dx$

解:$\int \frac{1+x+x^2}{x(1+x^2)}dx = \int \frac{(1+x^2)+x}{x(1+x^2)}dx = \int \frac{1}{x}dx + \int \frac{1}{1+x^2}dx = \ln|x| + \arctan x + C$

例 2、求 $\int \frac{1}{x^2-a^2}dx$

解:$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\int (\frac{1}{x-a} - \frac{1}{x+a})dx = \frac{1}{2a}[\int \frac{1}{x-a}d(x-a) - \int \frac{1}{x+a}d(x+a)]$

$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

例 3、求 $\int \frac{x+3}{x^2-5x+6}dx$

解：因为 $\frac{x+3}{x^2-5x+6} = \frac{x+3}{(x-2)(x-3)} = \frac{-5}{x-2} + \frac{6}{x-3}$

所以 $\int \frac{x+3}{x^2-5x+6}dx = \int \left(\frac{-5}{x-2} + \frac{6}{x-3}\right)dx = -5\int \frac{1}{x\text{-}2}dx + 6\int \frac{1}{x-3}dx$

$= \text{-}5\ln|x\text{-}2| + 6\ln|x-3| + C$

例 4、求 $\int \frac{x-2}{x^2+2x+3}dx$

解：由于分母已为二次质因式，分子 $= x-2 = \frac{1}{2}(2x+2)-3$，于是

$\int \frac{x-2}{x^2+2x+3}dx = \int \frac{\frac{1}{2}(2x+2)-3}{x^2+2x+3}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx - 3\int \frac{dx}{x^2+2x+3}$

$= \frac{1}{2}\int \frac{d(x^2+2x+3)}{x^2+2x+3} - 3\int \frac{d(x+1)}{(x+1)^2+(\sqrt{2})^2} = \frac{1}{2}\ln(x^2+2x+3) - \frac{3}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$

例 5、求 $\int \frac{1}{(1+2x)(1+x^2)}dx$

解:$\int \frac{1}{(1+2x)(1+x^2)}dx = \int \frac{\frac{4}{5}}{1+2x} + \frac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2}dx = \frac{2}{5}\int \frac{2}{1+2x}dx - \frac{1}{5}\int \frac{2x}{1+x^2}dx + \frac{1}{5}\int \frac{1}{1+x^2}dx$

$= \frac{2}{5}\int \frac{1}{1+2x}d(1+2x) - \frac{1}{5}\int \frac{1}{1+x^2}d(1+x^2) + \frac{1}{5}\int \frac{1}{1+x^2}dx$

$= \frac{2}{5}\ln|1+2x| - \frac{1}{5}\ln(1+x^2) + \frac{1}{5}\arctan x + C$

例 6、 $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$

解：如果作变量代换 $u=\tan\frac{x}{2}$，由万能公式得

$$\sin x=\frac{2u}{1+u^2}，\ \cos x=\frac{1-u^2}{1+u^2}，\ dx=\frac{2}{1+u^2}du$$

于是$\int \frac{1+\sin x}{\sin x(1+\cos x)}dx=\int \frac{(1+\frac{2u}{1+u^2})}{\frac{2u}{1+u^2}(1+\frac{1-u^2}{1+u^2})}\frac{2}{1+u^2}du=\frac{1}{2}\int (u+2+\frac{1}{u})du$

$$=\frac{1}{2}(\frac{u^2}{2}+2u+\ln|u|)+C=\frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\ln|\tan\frac{x}{2}|+C$$

例7、 求 $\int \frac{dx}{1+\sqrt[3]{x+2}}$

解：令 $\sqrt[3]{x+2}=u$，得 $x=u^3-2$， $dx=3u^2du$，故

$$\int \frac{dx}{1+\sqrt[3]{x+2}}=\int \frac{3u^2}{1+u}du=3\int \frac{u^2-1+1}{1+u}du=3\int (u-1+\frac{1}{1+u})du$$

$$=3(\frac{u^2}{2}-u+\ln|1+u|)+C=\frac{3}{2}\sqrt[3]{(x+2)^2}-3\sqrt[3]{x+2}+3\ln|1+\sqrt[3]{x+2}|+C$$

例8、 求 $\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$

解： 令 $x=t^6$，得 $dx=6t^5dt$，于是

$$\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}=\int \frac{6t^5dt}{(1+t^2)t^3}=6\int \frac{t^2}{1+t^2}dt=6\int (1-\frac{1}{1+t^2})dt$$

$$=6(t-\arctan t)+C=6(\sqrt[6]{x}-\arctan\sqrt[6]{x})+C$$

5、递推法：例求$\int \sin^{10}xdx$

$$\int \sin^{10}xdx=-\int \sin^9 xd\cos x=-\sin^9 x\cos x+9\int \cos^2 x\sin^8 xdx$$

$$=-\sin^9 x\cos x+9\int \sin^8 xdx-9\int \sin^{10}xdx$$

于是$\int \sin^{10}xdx=-\frac{1}{10}\sin^9 x\cos x+\frac{9}{10}\int \sin^8 xdx$

6、一题多解

求 $\int \sec x dx$

解1: $\int \sec x dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{1}{1-\sin^2 x} d\sin x = \frac{1}{2}\int (\frac{1}{1+\sin x} + \frac{1}{1-\sin x}) d\sin x$

$= \frac{1}{2}(\ln|1+\sin x| - \ln|1-\sin x|) + C = \frac{1}{2}\ln|\frac{1+\sin x}{1-\sin x}| + C = \frac{1}{2}\ln|\frac{(1+\sin x)^2}{\cos^2 x}| = \ln|\frac{1+\sin x}{\cos x}|$

解2: $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$

$= \ln|\sec x + \tan x| + C$

注: $\frac{1}{2}\ln|\frac{1+\sin x}{1-\sin x}| + C = \frac{1}{2}\ln|\frac{(1+\sin x)^2}{\cos^2 x}| = \ln|\frac{1+\sin x}{\cos x}| = \ln|\sec x + \tan x|$

解3: 设 $\tan\frac{x}{2} = t, \frac{1}{2}\sec^2\frac{x}{2} dx = dt, dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+\tan^2\frac{x}{2}} = \frac{2dt}{1+t^2}, \sec x = \frac{1}{\cos x} = \frac{1+t^2}{1-t^2}$

$\int \sec x dx = \int \frac{2dt}{1-t^2} = \int (\frac{1}{1+t} + \frac{1}{1-t}) dt = \ln|\frac{1+t}{1-t}| + C = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$

注: $\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \frac{1+\sin x}{\cos x} = \sec x + \tan x$